Numerical Computing

Semestral Examination

May 04, 2026

Instructions: All questions carry ten marks. Duration is three hours

1. Let A be an $m \times n$ real matrix. Prove that the rank of A equals the rank of A^tA, where A^t is the transpose of A.

2. Define a *natural cubic spline* on an interval $[a, b]$. Derive the equations for the natural cubic spline that fits the following table:

x	-1	0	1
y	1	2	-1

3. Derive the Newton's formula for finding a root of a non-linear function. The reciprocal of a non-zero number R can be computed by the iterative formula

$$x_{n+1} = x_n(2 - Rx_n)$$

Establish this relation using the Newton's method for a suitable function.

4. Let n be an even number and assume that an interval $[a, b]$ is divided in n equal subintervals of length h with end points $x_0, x_1, \ldots x_n$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then prove that the composite Simpson's rule for computing $\int_a^b f(x)dx$ is given by the formula:

$$\frac{h}{3}[f(x_0) + f(x_n) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + 2\sum_{1}^{\frac{n-2}{2}} f(x_{2i})]$$

5. Prove that for each $n \geq 1$, there exists a unique polynomial $q_n(x)$ such that

(a) $q_n(1) = 1$, and

(b) $\int_{-1}^{1} p(x)q_n(x)dx = 0$ for every polynomial $p(x)$ of degree at most $n - 1$.

(Remark: These q_n's are called the Legendre Polynomials)